VIBRATION MODES OF AN ORBITING RADIO TELESCOPE

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SUMMARY

The vibration mode shapes and frequencies of a 1500-meter orbiting radio telescope are calculated and discussed. The radio telescope employs a paraboloidal reflector constructed from thin aluminum tapes arranged in a gridwork of equilateral triangles. The shape of the reflector is maintained by spin about its axis of symmetry. A central column carries the resulting compressive load.

The sources of dynamic loads in a space environment are discussed and the general requirements for dynamic resonance in a spinning elastic body are developed. The method employed for calculating vibration modes is described. Discussions of the effects of initial curvature and static preload on stiffness are included.

It is shown that, for the configuration analyzed, potentially important resonances with applied loads may exist for several harmonic orders of the rotational speed.

INTRODUCTION

The analysis described in this report is a part of an investigation of the feasibility of a large orbiting radio telescope. Other aspects of the investigation are described in references 1 to 7.

The radio telescope employs a paraboloidal reflector and is designed to have a half-power beam width of three degrees at four megacycles. For this reason it is necessarily of very large size. The main features and the dimensions of the radio telescope are shown in figure 1.
Static and dynamic distortions of the reflector are important design considerations because of the aberrations they produce in the image of a radiating source. The main objective in the dynamic analysis of an orbiting telescope is the calculation of the deflections of the reflector due to the several sources of dynamic loads that are present in a space environment. The calculation of vibration modes is an important first step in the determination of dynamic response.

The report contains a brief description of the dynamic loads environment, a theoretical discussion of the methods used in vibration analysis of the antenna, and a presentation of calculated vibration mode shapes and frequencies. It also contains, as introductory material, a description of the structural design features of the antenna that are important for dynamic analysis.

SYMBOLS

A  cross-sectional area
a_{jk}  differential operator with \( r \) and \( z \) as independent variables
B  damping matrix
E  Young's modulus
E_{eff}  effective extensional modulus
f  inertia force
I_p  moment of inertia about spin axis
I_x  moment of inertia about transverse axis
i = \sqrt{-1}
K  stiffness matrix; spring constant
L(u)  a general differential operator
\( l_C \) length of tape with one crease

\( l_1 \) length of a spiral member

\( M \) mass matrix

\( m \) mass per unit area

\( m_c \) mass of column per unit length

\( N_S \) meridional static force density (lb/unit length)

\( N_\varphi \) circumferential static force density (lb/unit length)

\( n \) number of circumferential waves

\( o(\ ) \) order of magnitude of

\( P = \alpha + i\omega \)

\( P \) load vector

\( q_C \) forces of constraint

\( R \) matrix of constraint coefficients

\( R^T \) transpose of \( R \)

\( r \) radius

\( s \) distance along meridian

\( T_{\text{spin}} \) period for one revolution

\( T_{\text{orb}} \) orbital period

\( t \) tape thickness; time

\( V \) potential energy

\( V_P \) phase velocity

\( u \) displacement vector

\( u_d \) dependent components of displacement
\( u_i \)  
independent components of displacement

\( u_s \)  
displacement in meridional direction

\( u_\varphi \)  
displacement in circumferential direction

\( w \)  
displacement normal to surface

\( X \)  
impedance matrix

\( y_i \)  
generalized displacement

\( z \)  
distance parallel to axis of symmetry

\( \alpha \)  
exponential decay factor

\( \beta \)  
angle between meridian and plane normal to the axis of symmetry

\( \beta' = \frac{\partial \beta}{\partial s} \)  
angle between spiral fiber and meridian

\( \Delta \)  
increment operator

\( \delta T \)  
kinetic energy per unit area

\( \delta V_e \)  
elastic strain energy per unit area

\( \delta V_s \)  
potential energy due to static preload, per unit area

\( \varepsilon_s \)  
extensional stress in meridional direction

\( \varepsilon_{s\varphi} \)  
membrane shear stress

\( \varepsilon_\varphi \)  
extensional stress in circumferential direction

\( \theta_s \)  
rotation about line tangent to meridian

\( \theta_w \)  
rotation about normal to surface

\( \theta_\varphi \)  
rotation about line tangent to polar circle

\( \sigma_{gg} \)  
stress due to gravity gradient

\( \sigma_\gamma \)  
axial stress in spiral tapes
An overall view of the radio telescope is shown in figure 1. The telescope consists of the following major components:
- Reflector Grid
- Rim Mass
- Central Compression Column
- Front Tensioning Network
- Back Tensioning Network
- Equipment Package at Forward End of Column
- Equipment Package at Aft End of Column

The weight of each of the major components is listed in table I.

The telescope spins about its axis of symmetry at the rate of approximately one revolution in 16 minutes. The resulting centrifugal force produces tension in the fibers of the reflector grid and compression in the central column.

The reflector grid consists of thin tapes of aluminum foil arranged as a network of equilateral triangles. Mechanical properties of the tapes are listed in table II. The spacing and cross-sectional area of the tapes were selected to meet the requirement for adequate electromagnetic reflectivity in the frequency range from one to ten megacycles, (ref. 1). The choice of very thin and relatively wide tapes rather than solid wires or hollow tubes was made from consideration of fracture
rates due to micrometeoroids (ref. 6), and from consideration of the requirement for foldability during packaging and deployment of the radio telescope. Commercially pure aluminum was selected as the material for the reflector grid due to its availability in very thin sheets and its low yield stress. The latter property minimizes the shortening of the tapes due to creases (ref. 3) and also increases the effective stiffness of the reflector grid, as will be shown.

The rim mass at the outer radius of the reflector is required in order to equilibrate the tensile forces in the reflector and in the front tensioning network by means of centrifugal force.

A similar mass is not required at the inner radius of the reflector because the back tensioning network is tangent to the reflector. The rim mass consists, largely, of insulated electrical conductors. Direct current flowing in these conductors interacts with the Earth's magnetic field to produce a precession- al control torque on the antenna, (ref. 5).

The central compression column is constructed as a lattice column of constant cross section with three tubular longerons and cross-ties. The column is designed to deploy automatically and continuously, (ref. 4). The bending stiffness of the central compression column was selected to prevent buckling and to provide a critical speed for lateral deflection that is well above the operating range.

The front and back tensioning networks consist of fiberglass tapes. Cross-ties, not shown in figure 1, will be required to maintain the conical shape of the tensioning networks and to minimize mass unbalance caused by tapes that have been broken by meteoroids. The mass of the tensioning networks has been neglected in the analysis described in this report.

The front equipment package consists of a dipole array for feeding the radio telescope, and electronic equipment for signal conditioning and communication. The back equipment package consists of solar cells and power supplies for the supply of current to the rim conductors and to electronic equipment.

The static stress distribution induced in the structural members of the radio telescope by centrifugal force is shown in figure 2. $\sigma_y$ is the stress in the spiral reflector tapes and
\( \sigma_\circ \) is the stress in the circumferential reflector tapes. It will be noted that the magnitudes of the stresses are very low compared to conventional engineering experience. It would be desirable to have larger stresses in order to reduce the shortening of the tapes due to initial curvature and creases. Higher stresses would, however, require a heavier central compression column and would, due to the higher rotational speed, increase the power required by the scanning and orientation control system.

The effective stiffness of tapes in the reflector grid was calculated on the assumption that each member of the grid has two sharp creases. The actual number of creases per member depends on the detailed design of the joints in the gridwork and on the manner of packaging. The following formula is given in reference 3 for the effective extensional modulus of a tape with initial transverse creases.

\[
\frac{1}{E_{\text{eff}}} = \frac{1}{E} + \frac{t}{2t_c} \left( \frac{E \sigma_y}{\sigma_\circ} \right)^{\frac{1}{2}} \cdot f \left( \frac{\sigma t E}{\sigma_\circ^2} \right)
\]

where

- \( E \) = Young's modulus
- \( t \) = thickness of tape
- \( t_c \) = length of tape with one crease
- \( \sigma t \) = working axial stress in the tape
- \( \sigma_y \) = yield stress of the material
- \( f \left( \frac{\sigma t E}{\sigma_\circ^2} \right) \) = function tabulated in figure 3.

It will be noted that if the second term in equation (1) predominates, low Young's modulus, high axial stress, low yield stress and small tape thickness all tend to increase the effective extensional stiffness. Effective extensional moduli for the spiral and circumferential members of the reflector grid are plotted in figure 4. The generally low values, particularly for the spiral members will be noted.

It has been determined that creases in the front and back tensioning networks can probably be avoided. Initial curvature will, however, tend to reduce the effective stiffness. The effective extensional modulus of the members in these networks
is estimated to be at least 50,000 psi.

**DYNAMIC LOADS ENVIRONMENT**

Static and dynamic distortions of the radio telescope are important design considerations because of the aberrations they produce in the image of a radiating source. The stresses produced by environmental loads are not likely to be significant from the viewpoint of structural integrity, with the possible exception of fatigue stresses in sharp creases. The permissible displacements of points in the reflector grid are relatively large. If it is assumed that a distortion equal to one-sixteenth of a wavelength is acceptable, then the permissible displacement from a paraboloidal surface at a frequency of 10 megacycles is about 6.5 ft or about 0.0013 times the diameter of the reflector.

Due to the spin of the antenna about its axis, environmental loads that would normally be considered to be static for a non-rotating body produce dynamic effects that are subject to resonant amplification. A static load distribution in a non-rotating coordinate system becomes a traveling wave in a coordinate system that rotates with the antenna. Details of the transformation between the non-rotating and rotating coordinate systems and identification of the important parameters for resonant amplification are discussed in the next section.

No attempt is made in this report to determine the magnitudes of the distortions due to dynamic loads, since this phase of the dynamic analysis of the orbiting radio telescope has not yet begun. Order-of-magnitude estimates of the more obvious sources of dynamic loads are made below in order to indicate the need for dynamic analysis.

**Gravity Gradient**

A direct effect of gravity gradient upon an orbiting body is to produce tension along an axis pointing toward the center of the Earth and to produce compression along transverse axes. If we consider an axisymmetric body rotating about an axis perpendicular to the plane of its orbit, the variation of compressional load around its circumference will contain primarily zero and second harmonic components. It is shown in reference 8 that
the order of magnitude of the stresses due to gravity gradient is smaller than the stresses due to spin by the ratio of the square of the rotational period to the square of the orbital period. Thus

\[
\frac{\sigma_{gg}}{\sigma_{\text{spin}}} = 0 \left( \frac{T_{\text{spin}}}{T_{\text{orb}}} \right)^2
\]  

(2)

For the present example, \( T_{\text{spin}} = 16 \text{ min} \) and \( \sigma_{\text{spin}} \approx 10 \text{ psi} \). The design altitude of the orbit is 6000 kilometers above the surface of the Earth. At this altitude the orbital period is approximately 235 minutes. Thus

\[
\sigma_{gg} \approx 10 \times \left( \frac{16}{225} \right)^2 = 0.05 \text{ psi}
\]  

(3)

Assuming an effective modulus for the tapes in the reflector equal to 1000 psi, the order of magnitude of their strain is \( \varepsilon = 0.05 \times 10^{-3} \). Barring large amplification due to dynamic resonance or mechanical leverage, it would appear that the distortions produced by gravity gradient are small compared to the permitted value of 0.0013 times reflector diameter.

Thermal Expansion

The heat flux into the tapes due to solar radiation varies over the surface of the reflector because of differences in the orientation of the tapes with respect to the sun. The resulting temperature variation is shown in figure 5 for a case in which the direction of illumination is perpendicular to the axis of the antenna and the ratio of solar absorptivity to emissivity is 2.5. For this case the largest temperature variation occurs in the circumferential tapes of the reflector. The minimum temperature of the tapes is established by radiation from the Earth's surface. Using the maximum temperature difference (500 F) shown in figure 5 and the thermal expansion coefficient of aluminum (13.3 \( \times 10^{-6} /{ }^\circ\text{F} \)), the resulting maximum variation in strain is 0.0066. If the ratio of the resulting normal displacement to reflector diameter were the same size, thermal distortion would probably produce significant aberration at 10 megacycles.
If thermal expansion proves to be a significant problem, its effects can be reduced by coating the tapes with a material (such as gold) with a lower ratio of solar absorptivity to emissivity.

**Photon Pressure**

Solar radiation also produces a direct mechanical force on the elements of the reflector due to photon pressure. Using a value of \(0.9 \times 10^{-6}\) dynes/cm\(^2\) for the photon pressure, the total load on the reflector, assuming full illumination on the entire surface, is about 0.03 lbs, which is about 1/15 of the axial component of tension in the spiral fibers due to spin. Assuming the ratio of fiber stress to axial load to be the same for the two conditions, the resulting stresses in the reflector gridwork are of the order of 0.5 psi, or ten times the stresses due to gravity gradient. The corresponding strains, assuming an effective modulus of 1000 psi, are of the order of 0.0005, or about 1/10 of the strains due to thermal expansion. Differences in the orientations of tapes with respect to the sun will produce variations in loading over the surface.

**Mass Unbalance**

Mass unbalance will exist due to imperfections of construction including initial curvature of the central column, and due to changes in the position of tapes fractured by micrometeoroids. Resonant amplification of the resulting unbalanced loading can easily be avoided by designing the central column to have a critical speed well above the operating range. Is not expected to be a serious problem.

**Transient Control Forces**

As mentioned earlier, the torque required to precess the spin axis of the antenna is produced by interaction of the Earth's magnetic field with current flowing in the rim. Sudden application or reversal of the torque will produce transient motions of the reflector. Design studies, ref. 5, indicate that a maximum torque capability of 100 ft-lbs is adequate to produce the required scanning rate (180 degrees per day). A sudden reversal of this torque would produce oscillatory stresses in the spiral tapes of the reflector grid of the order of 0.4 psi which is approximately
the same size as the stresses produced by photon pressure. Distortions of the reflector due to transient control forces can therefore be expected to be of the order of .0005 times the diameter of the reflector.

METHOD OF ANALYSIS

General Theory for the Vibrations of a Rotating Axisymmetric Body

The homogeneous equations of small motion for an elastic axisymmetric body can be written as partial differential equations in a cylindrical coordinate system with \( r, z, \varphi, \) and \( t \) as independent variables, such that the coefficients of the terms are functions of \( r \) and \( z \) only. We wish first to show the manner in which the solution depends on \( \varphi \) and \( t \). For this purpose, let us examine a typical equation of equilibrium which may be written

\[
L(u) = \sum_{j,k} a_{jk} \frac{\partial^{j+k} u}{\partial \varphi^j \partial t^k} = 0
\]  

(4)

where \( a_{jk} \) is a differential operator with respect to \( r \) and \( z \), and \( u \) is the displacement vector.

A general solution that satisfies equation (4) is

\[
u = \bar{u}(r,z)e^{(pt+b\varphi)}
\]

(5)

where \( p \) and \( b \) are constants independent of \( r, z, \varphi \) or \( t \). Substitution into equation (4) gives

\[
L(u) = \left[ \sum_{j,k} a_{jk} b^j p^k \right] \bar{u} e^{(pt+b\varphi)} = 0
\]

(6)

which, since the exponential term may be factured out, results in a partial differential equation with dependence on \( r \) and \( z \) only.
A requirement on the solution is that it be single-valued throughout the body, for which a necessary condition is that the dependence on $\Phi$ be periodic with period equal to $2\pi$. This condition is satisfied if $b = in$ where $n$ is a (positive or negative) integer. Let the parameter $p$ be represented by its real and imaginary parts, $p = \alpha + i\omega$. The general solution, equation (5), may then be written

$$u = u(r,z)e^{\alpha t}e^{i(\omega t+n\Phi)}$$ (7)

Physically the solution given by equation (7) represents an exponentially increasing wave traveling in the negative $\Phi$ direction with phase velocity $V_p = \frac{\omega}{n}$ (8)

Since the constants $\omega$ and $n$ may be negative as well as positive, waves may also travel in the forward $\Phi$ direction.

For the case of undamped vibrations, $\alpha = 0$, substitution of the general solution, equation (7), into equation (4) results in the following eigenvalue problem:

$$\sum_{j,k} a_j n^j (i\omega)^k \tilde{u} = 0$$ (9)

For each value of $n$, a series of values of $\omega$ ($\omega_{n_1}, \omega_{n_2}, \ldots$) may be found for which non-trivial solutions ($\tilde{u}_{n_1}, \tilde{u}_{n_2}, \ldots$) exist that satisfy equation (9) and its boundary conditions. $\omega_{n_i}$ and $\tilde{u}_{n_i}$ are respectively the frequency and mode shape of natural vibration modes of the structure. Equation (9) is an important result for numerical analysis because it shows that the three-dimensional vibration problem can be reduced to a series of simpler two-dimensional vibration problems.

The homogeneous solutions to the general vibration problem of an axisymmetric body have been shown to be traveling waves. It is of interest to examine the conditions under which a
Standing wave is a possible homogeneous solution. A standing wave is represented by the form

\[ u = \bar{u}(r,z)e^{i\omega t}\cos(n\psi) \]

\[ = \frac{1}{2} \bar{u}(r,z)\left[ e^{i(\omega t+n\psi)} + e^{i(\omega t-n\psi)} \right] \] (10)

Clearly standing waves exist as solutions if, and only if, eigenvalues \( \omega_n \) and eigenvectors \( \bar{u}_n \) can be found that are the same when the sign of \( n \) is changed. Examination of equation (9) shows that this condition exists if all values of \( j \), the order of differentiation with respect to \( \varphi \), are even, or if all values of \( j \) are odd. For the case of a spinning body it can be shown that the inertia forces, when expressed in a non-rotating coordinate system, give rise to terms of mixed odd and even order in the differential equations so that standing wave solutions do not exist, in general, for spinning bodies. Standing wave solutions do, of course, exist for non-rotating axisymmetric bodies.

Consider next the response of a spinning body to an applied load distribution that is stationary in time and space with respect to a non-rotating reference frame. The load distribution in the non-rotating reference frame may be expanded in a Fourier series in \( \varphi \)

\[ P = P(r,z,\tilde{\varphi}) = \sum_n a_n \sin(n\tilde{\varphi}) + b_n \cos(n\tilde{\varphi}) \] (11)

where the coefficients \( a_n \) and \( b_n \) are functions of \( r \) and \( z \). \( \tilde{\varphi} \) is the azimuth position in the non-rotating system and is related to \( \varphi \), the azimuth position in the rotating system as shown in the following diagram.
From the diagram

$$\Phi = \varphi + \Omega t$$

(12)

where $\Omega$ is the angular velocity of rotation.

Thus, substituting into equation (11)

$$p = \sum_{n} a_n \sin(n\Omega t + n\varphi) + b_n \cos(n\Omega t + n\varphi)$$

$$= (\text{Real Part of}) \sum_{n} c_n e^{i(n\Omega t + n\varphi)}$$

(13)

The applied load is therefore represented in the rotating coordinate system by a series of backward traveling waves with frequencies at the harmonics of the rotational speed. The $n$th harmonic component of load will excite only the vibration modes associated with backward traveling waves of the same harmonic order. Resonant amplification will occur for modes that have natural frequencies near $n\Omega$. An equivalent statement, obtained from equation (8) is that resonant amplification will
occur for modes in which the phase velocity relative to a non-
rotating coordinate system is near zero.

Since most of the environmental loads acting on the orbiting
radio telescope are stationary in a non-rotating reference frame,
interest in the analysis of free vibration modes will center on
the closeness to resonance of waves that are traveling backward
relative to a rotating reference frame.

It is of interest to note that the observed frequencies of
vibration are different in the rotating and non-rotating systems.
The reason is that, from equation (12)

$$\omega t + n\varphi = (\omega - n\Omega)t + n\varphi$$

(14)

The frequency for waves that travel backward in the rotating
coordinate system is, therefore, lowered by $n\Omega$ when observed
in the non-rotating coordinate system. The frequency for forward
traveling waves is increased by $n\Omega$ when observed in the non-
rotating coordinate system. Equation (14) also shows that the
frequency for resonance with stationary applied loads is zero in
the non-rotating coordinate system.

Equations for the Reflector

Displacements of the reflector are represented by orthogonal
components normal and tangential to the surface of the reflector
in a rotating coordinate system as shown in figure 6. Let the
motions of the reflector correspond to an $n^\text{th}$ harmonic backward
traveling wave as indicated by the following equations

$$\begin{bmatrix}
  u_s \\
  u_\varphi \\
  w
\end{bmatrix} = 
\begin{bmatrix}
  \text{Real Part of} \\
  \text{Part of}
\end{bmatrix}
\begin{bmatrix}
  u_{sn} \\
  -iu_{\varphi n} \\
  w_n
\end{bmatrix} e^{(pt + i\varphi)} = 
\begin{bmatrix}
  u_{sn} e^{\alpha t} \cos(\omega t + n\varphi) \\
  -u_{\varphi n} e^{\alpha t} \sin(\omega t + n\varphi) \\
  w_n e^{\alpha t} \cos(\omega t + n\varphi)
\end{bmatrix}$$

(15)

where the substitution $p = \alpha + iw$ has been made. The choice
of an imaginary coefficient for $u_{\varphi n}$ is made in order to produce
equations with a minimum of imaginary coefficients.

The harmonic coefficients $u_{sn}$, $w_n$, and $\varphi_n$ are treated as degrees of freedom. The technique used in the solution employs, as a first step, the expression of the elastic strain energy and the potential energy due to static preload in terms of membrane strains and rotations. The membrane strains and rotations are expressed in terms of harmonic coefficients and have traveling wave dependence similar to that shown for the components of motion by equation (15). The following relationships between harmonic coefficients are derived in reference 9.

Strain-Displacement Relationships:

\begin{align*}
\varepsilon_{sn} &= \frac{\partial u_{sn}}{\partial s} - \beta' w_n \quad \text{(cosine)} \quad (16) \\
\varepsilon_{\varphi n} &= \frac{n}{r} u_{\varphi n} + \frac{1}{r} \left( u_{sn} \cos \beta - w_n \sin \beta \right) \quad \text{(cosine)} \quad (17) \\
\varepsilon_{s\varphi n} &= \frac{\partial u_{\varphi n}}{\partial s} - \frac{1}{r} \left( u_{\varphi n} \cos \beta - nu_{sn} \right) \quad \text{(sine)} \quad (18)
\end{align*}

Rotation-Displacement Relationships:

\begin{align*}
\theta_{sn} &= \frac{1}{r} \left( u_{\varphi n} \sin \beta - nw_n \right) \quad \text{(sine)} \quad (19) \\
\theta_{\varphi n} &= \frac{\partial w_n}{\partial s} + \beta' u_{sn} \quad \text{(cosine)} \quad (20) \\
\theta_{wn} &= \frac{1}{2} \left[ \frac{\partial u_{\varphi n}}{\partial s} + \frac{1}{r} \left( u_{\varphi n} \cos \beta + nu_{sn} \right) \right] \quad \text{(sine)} \quad (21)
\end{align*}

Potential energy is computed as follows for an annular strip of slant length $\Delta s$:

\[ \Delta V = \int_{s}^{s+\Delta s} \int_{0}^{2\pi} \left( \delta V_e + \delta V_s \right) r d\varphi ds \quad (22) \]
\( \delta V_e \) is the elastic strain energy per unit area and is a quadratic function of the strains. \( \delta V_s \) is the potential energy due to static preload per unit area and is a quadratic function of the rotations. An elementary analysis shows that the potential energy for the \( n \)th harmonic order is the following function of the harmonic strain and rotation coefficients.

\[
\Delta V_n = \frac{\pi r s}{2} \left[ \frac{(EA) \phi}{l_1 \cdot \cos \gamma} \cdot \phi_n^2 + \frac{(EA) \cdot \sin \gamma \cdot \cos \gamma}{l_1} \left( \frac{\epsilon_{sn} \cdot \cot \gamma + \epsilon_n \cdot \tan \gamma}{\epsilon_{sn} \cdot \cot \gamma + \epsilon_n \cdot \tan \gamma} + \phi_n^2 \right) + N_s \phi_n^2 + N \phi_n\omega_n + \left( N_s + N \right) \omega_n^2 \right] \tag{23}
\]

where

- \( (EA) \phi \) = effective axial stiffness of a circumferential fiber
- \( (EA) s \) = effective axial stiffness of a spiral fiber
- \( l_1 \) = length of a spiral member
- \( \gamma \) = angle between meridian and a spiral member
- \( N_s \) = meridional static force density (lb/ unit length)
- \( N \phi \) = circumferential static force density (lb/unit length)

The next step in the solution is to obtain the generalized inertia forces associated with the harmonic coefficients of the traveling wave solutions. For this purpose use is made of the following expression for the kinetic energy density ascribed to perturbation motions derived in reference 9.
where \( m \) is the mass per unit area.

Application of Lagrange's equations and introduction of the traveling wave solutions yields the following matrix equation for the generalized inertia forces acting on a strip of slant length \( \Delta s \)

\[
\begin{align*}
\begin{bmatrix}
    f_{sn} \\
    f_{\varphi n} \\
    f_{wn}
\end{bmatrix} &= - m r \Delta s 
\begin{bmatrix}
    p^2 - \Omega^2 \cos^2 \beta & 2i\Omega p \cos \beta & \Omega^2 \sin \beta \cos \beta \\
    2i\Omega p \cos \beta & p^2 - \Omega^2 & 2i\Omega p \sin \beta \\
    \Omega^2 \sin \beta \cos \beta & -2i\Omega p \sin \beta & p^2 - \Omega^2 \sin^2 \beta
\end{bmatrix} 
\begin{bmatrix}
    u_{sn} \\
    u_{\varphi n} \\
    w_{n}
\end{bmatrix}
\end{align*}
\]

(25)

Note that the Coriolis force terms appear as a symmetric matrix of imaginary damping coefficients, and that reversal of the direction of rotation, which is equivalent to changing the sign of \( n \), changes the sign of these terms. Hence the solutions for positive and negative \( n \) will be different.

The complete homogeneous equations of equilibrium for an annular strip of the reflector are obtained by taking the partial derivatives of the potential energy (equation 23) with respect to the harmonic coefficients \( u_{sn}, u_{\varphi n}, \) and \( w_{n} \) and equating them to the generalized inertia forces. Nothing is served by carrying out this operation, however, because the computer program used for solving the resulting eigenvalue problem readily accepts information in a form more closely identified with equation (23) and the relationships shown in equations (16) to (21).
Problem Formulation for Computer Solution

The method of analysis was tailored to an available digital computer program, SADSAM IV, developed by the MacNeal-Schwendler Corporation. A brief account of the operating features of the program is important for an understanding of the approach taken.

SADSAM IV is a program of modest capacity (50 independent degrees of freedom) for the solution of structural dynamic problems (eigenvalue extraction, transient analysis and frequency response analysis) by the lumped element approach. A significant feature of the program is the employment of equations of constraint between dependent and independent displacement components. The general matrix equation solved by the program can be written as follows:

\[
\begin{bmatrix}
X_{ii} & X_{id} & -R^T_i \\
X_{di} & X_{dd} & -R^T_d \\
R_i & R_d & 0
\end{bmatrix}
\begin{bmatrix}
u_i \\
u_d \\
q_c
\end{bmatrix} =
\begin{bmatrix}
p_i \\
p_d \\
0
\end{bmatrix}
\]

where

- \( u_i \) are independent components of displacement
- \( u_d \) are dependent components of displacement
- \( q_c \) are forces of constraint

The impedance matrix \( X_{ii} \) consists of mass, damping and stiffness components

\[
X_{ii} = M_{ii}p^2 + B_{ii}p + K_{ii},
\]

and similarly for \( X_{id} \), \( X_{di} \), and \( X_{dd} \). The elements of matrices \( R_i \) and \( R_d \) are the coefficients of the equations of
The stiffness matrix $K_{ii}$ is assembled (in part) from the properties of simple springs connected between pairs of displacement components. In addition, the user can specify matrix elements to be inserted directly into $K_{ii}$, a feature that is used for the generation of aerodynamic force coefficients and other unconservative effects. $B_{ii}$ and $M_{ii}$ are assembled in similar fashion from simple dampers and masses, and from direct input.

The coefficients of the equations of constraint, $R_i$ and $R_d$ in equation (26), are specified by the user to express rigid body properties of the elements of the structure and to express coordinate transformations. They may also be used, as in the present example, to express strains and rotations in terms of displacements.

It will be noted that the potential energy of the reflector, equation (23), is expressed in the form

$$
\Delta V_n = \frac{1}{2} \sum_i K_{ii} y_i^2
$$

where $y_i$ is a generalized displacement quantity related to the harmonic components of displacement by a linear operator, (equations 16 - 21). Hence $K_{ii}$ represents a spring restraining $y_i$ and it is represented as such in the computer program. The relationship between each $y_i$ and the harmonic components of displacement is regarded as an equation of constraint. Differential operators, $\left( \frac{\partial}{\partial s} \right)$, occurring in the equation of constraint are replaced by differential operators, e.g.,

$$
\left. \frac{\partial w_n}{\partial s} \right|_{m+\frac{1}{2}} = \frac{w_{n,m+1} - w_{n,m}}{\Delta s}
$$

where the subscripts $m$ and $m+1$ refer to adjacent stations.
along the meridian and \( m^{1+\frac{1}{2}} \) is halfway between them. The terms proportional to \( \Omega^2 \) in the kinetic energy, equation (24), are treated in similar fashion, such terms being represented by negative springs.

The terms proportional to \( p^2 \) in the inertia force equation (equation 25) are represented by lumped masses attached to \( u_{sn} \), \( u_{\varphi n} \) and \( w_n \). The Coriolis force terms are inserted by the user directly into the damping matrix, since such terms do not correspond to the properties of real physical dampers and cannot, therefore, be treated as structural elements.

The motions of the reflector were simulated by harmonic coefficients at the stations labeled 1 to 6 in figure 2. The increment in radius between stations is uniform.

The degrees of freedom for the central column were simulated by harmonic coefficients at eleven equally spaced stations. Harmonic coefficients for \( n \geq 2 \) were omitted since such motions involve elastic distortion of the cross section of the column. The column was treated as a flexural beam loaded compressively. The compressive load gives rise to negative springs attached to coordinates representing the slope of the beam. The inertia force per unit length acting on the column, for lateral motion in a rotating coordinate system, is

\[
 f_{ci} = -m_c \left[ p^2 - 2i\Omega p - \Omega^2 \right] w_1
\]

where \( w_1 \) is the first harmonic coefficient of lateral motion and \( m_c \) is the mass per unit length.

The front and back tensioning networks were analyzed in terms of their potential energy due to elastic strain and static preload, and were simulated by simple springs and equations of constraint. The distributed inertia of the tensioning networks was ignored, thus permitting simulation of the networks by a minimum number of springs, (three per network).
RESULTS OF NUMERICAL CALCULATION

Normal mode frequencies and mode shapes of the radio telescope were computed for several values of \( n \) and for both forward and backward traveling waves. Mode frequencies for the lower modes are listed in table III for \( n = 1, 2, 4 \) and 8, and are plotted versus \( n \) in figure 7. Figure 7 also includes results for a case in which the elastic stiffness of the reflector tapes was arbitrarily increased by a factor of four. The mode shapes of selected modes are plotted in Figures 8 to 12.

The frequencies of the modes are normalized by dividing by the rotational speed, which is 0.00648 radians per second. The recorded frequencies are those which would be observed in the rotating coordinate system. To obtain the frequencies in the non-rotating system, add \( n \) cycles per revolution for forward traveling waves and subtract \( n \) cycles per revolution for backward traveling waves.

The presence of four rigid body modes (two translational modes and two rotational modes) is indicated in table III. Three of these modes are backward traveling waves with a frequency of one cycle per revolution in the rotating reference system, which corresponds to zero cycles per revolution in the non-rotating reference system. These modes are, therefore, rigid body modes in the ordinary sense. A fourth rigid body mode is a forward traveling wave with a frequency equal to 0.42 cycles/revolution. This mode corresponds to the nutation of a spinning rigid body. Its frequency in the rotating system is equal to \([\frac{I_p}{I_x} - 1]\) cycles per revolution.

The forward and backward column bending modes are seen from table III to have nearly the same frequencies in the non-rotating system. Due to mechanical coupling between the column and the reflector, the frequency for the lowest column mode is about five percent higher than the frequency that the column would have if its ends were simply supported. The mode shapes plotted in figures 8 and 9 indicate relatively small motions of the reflector in the column modes.

The critical resonant frequency of the column for mass
unbalance is at zero cycles per revolution in the rotating system. The column bending modes are well removed from this resonance.

Mode shapes for the three lowest backward traveling wave modes of the reflector are plotted in figures 10, 11, and 12 for \( n = 2 \). It is seen that these modes consist predominantly of motion normal to the surface of the reflector. The mode shapes for other values of \( n \) are quite similar.

Points on the "resonance line" in figure 7 are the frequencies for resonance with an applied load distribution that is stationary in the non-rotating system. The frequency of the lowest reflector mode crosses the resonance line between \( n = 2 \) and \( n = 3 \) and remains close to the resonance line for several higher harmonic orders. Significant resonant amplification of these modes can therefore be expected.

An increase in the effective elastic stiffness of the reflector tapes significantly increases the mode frequencies for the lower harmonic orders but not for the higher harmonic orders. The reason is that, as the harmonic order is increased, the stiffness due to static preload becomes larger for motion normal to the surface of the reflector.

CONCLUDING DISCUSSION

The calculation of the normal modes of vibration of a large orbiting paraboloidal antenna has been described. Many of the features of the calculation are different from those encountered in the vibration analysis of ordinary terrestrial objects.

Among the differences are the following:

1. Bending stiffness of the members in the reflector is entirely negligible, except for an indirect effect that it has on extensional stiffness.

2. The extensional stiffness of the members of the reflector is extremely low due to the very low value of static stress that is available for straightening the members.
3. Differential stiffness due to static preload (tension in the reflector and compression in the central column) cannot be ignored.

4. The normal modes are traveling waves rather than standing waves.

5. The primary source of dynamic excitation is a distribution of loads that is stationary with respect to a non-rotating reference frame.

The results of the calculation show the existence of a potentially serious resonance condition for all harmonic orders higher than the second. Resonance can be avoided for the lower harmonic orders but not for the higher ones by increasing the extensional stiffness of the reflector.

Uncertainty with regard to the magnitude of the effective extensional stiffness of the reflector makes it impossible to predict accurately the crossings of the resonance line in figure 7 by the vibration modes. Therefore the possibility of a very near resonance in one of the higher harmonic orders must be accepted. Such resonance can be avoided, once its presence is detected, by a small change in the rotational speed.

Unfinished business for future investigation includes, principally, the calculation of the response to various sources of dynamic excitation, some of which have been described briefly in this report. It also includes the estimation of structural damping for use in the calculation of resonant response, and the investigation of the effects of configuration changes.

Astro Research Corporation
Santa Barbara, California, November 30, 1966.
REFERENCES

### TABLE I

**WEIGHT SUMMARY**

<table>
<thead>
<tr>
<th>Component</th>
<th>Weight (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflector Grid</td>
<td>1103</td>
</tr>
<tr>
<td>Rim Mass</td>
<td>472</td>
</tr>
<tr>
<td>Central Column</td>
<td>500</td>
</tr>
<tr>
<td>Front Equipment Package</td>
<td>220</td>
</tr>
<tr>
<td>Back Equipment Package</td>
<td>308</td>
</tr>
<tr>
<td>Front Tensioning Network</td>
<td>170</td>
</tr>
<tr>
<td>Back Tensioning Network</td>
<td>85</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>2858</strong></td>
</tr>
</tbody>
</table>

Polar Moment of Inertia, \( I_\text{p} \) \( = \ 225.8 \times 10^6 \) slug-ft²

Moment of Inertia about transverse axis, \( I_x \) \( = \ 158.5 \times 10^6 \) slug-ft²
### Reflector Grid:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer radius</td>
<td>2461 ft</td>
</tr>
<tr>
<td>Inner radius</td>
<td>1477 ft</td>
</tr>
<tr>
<td>Number of spiral tapes in one set</td>
<td>5100</td>
</tr>
<tr>
<td>Tape spacing at rim</td>
<td>36.4 in</td>
</tr>
<tr>
<td>Tape width</td>
<td>0.1 in</td>
</tr>
<tr>
<td>Tape thickness</td>
<td>.0005 in</td>
</tr>
<tr>
<td>Tape material</td>
<td>commercially pure aluminum</td>
</tr>
<tr>
<td>Yield stress</td>
<td>5000 psi</td>
</tr>
</tbody>
</table>

### Tensioning Networks:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of tapes</td>
<td>510</td>
</tr>
<tr>
<td>Tape width</td>
<td>0.1 in</td>
</tr>
<tr>
<td>Tape thickness</td>
<td>.001 in</td>
</tr>
<tr>
<td>Material</td>
<td>fiberglass</td>
</tr>
</tbody>
</table>

### Central Column:

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>EI</td>
<td>$1.1 \times 10^6$ lb-ft$^2$</td>
</tr>
</tbody>
</table>
### TABLE III

**VIBRATION-MODE FREQUENCIES**

#### (a) Mode Type

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Frequency (cycles/revolution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Backward Traveling Waves</td>
<td></td>
</tr>
<tr>
<td>Rigid Body Translation (2 modes)</td>
<td>1.0</td>
</tr>
<tr>
<td>Rigid Body Rotation</td>
<td>1.0</td>
</tr>
<tr>
<td>First Column Bending</td>
<td>3.54</td>
</tr>
<tr>
<td>Second Column Bending</td>
<td>12.02</td>
</tr>
<tr>
<td>First Reflector Mode</td>
<td>5.11  2.93  3.80  6.95</td>
</tr>
<tr>
<td>Second Reflector Mode</td>
<td>8.38  6.84  6.07  8.17</td>
</tr>
<tr>
<td>Third Reflector Mode</td>
<td>11.23 9.75  8.62  9.76</td>
</tr>
</tbody>
</table>

#### (b) Mode Type

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Frequency (cycles/revolution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward Traveling Waves</td>
<td></td>
</tr>
<tr>
<td>Rigid Body Rotation</td>
<td>0.42</td>
</tr>
<tr>
<td>First Column Bending</td>
<td>1.58</td>
</tr>
<tr>
<td>Second Column Bending</td>
<td>10.02</td>
</tr>
<tr>
<td>First Reflector Mode</td>
<td>4.58  2.58  3.63  6.89</td>
</tr>
<tr>
<td>Second Reflector Mode</td>
<td>8.16  6.69  5.95  8.09</td>
</tr>
<tr>
<td>Third Reflector Mode</td>
<td>11.20 9.64  8.50  9.69</td>
</tr>
</tbody>
</table>
Rotational Speed = 0.00648 rad/sec
Column Load = 0.458 lb

\[ \sigma_f = 14.95 \text{ psi} \]

\[ \sigma_Y = 2.45 \text{ psi} \]
\[ \sigma_\varphi = 21.98 \text{ psi} \]

\[ \tau = 2.84 \]
\[ \sigma = 13.04 \]

\[ \sigma_Y = 3.12 \]
\[ \sigma_\varphi = 9.0 \]

\[ \sigma_Y = 3.46 \]
\[ \sigma_\varphi = 5.15 \]

\[ \sigma_b = 31.4 \text{ psi} \]

Figure 2. — Static Stress Distribution due to Rotation
Figure 3. — Shortening of a Tape due to an Initial Crease
Figure 4. — Effective Modulus Distribution for Fibers
Figure 5. Temperature Distribution for Side Illumination
Figure 6. Coordinate Geometry for Reflector
Figure 7. — Vibration-Mode Frequencies of Reflector for Backward Traveling Waves
Figure 8. — First Column Bending Mode for Backward Traveling Waves
($\omega/\Omega = 3.54$ in the rotating system)
Figure 9. — Second Column Bending Mode for Forward Traveling Waves

\( \omega/\Omega = 10.02 \) in the rotating system
Figure 10. — Lowest Backward Mode
(n = 2, \( \omega = 0.01897 \text{ rad/sec} \), 
\( \omega/\Omega = 2.93 \text{ cycles/rev in the rotating system} \))
Figure 11. — Second Backward Mode

\( n = 2, \ \omega = 0.04428 \text{ cycles/sec}, \ \omega/\Omega = 6.84 \text{ cycles/rev in the rotating system} \)
Figure 12. — Third Backward Mode

\( n = 2 \), \( \omega = 0.06309 \text{ rad/sec} \)
\( \omega/\bar{u} = 9.75 \text{ cycles/rev in the rotating system} \)